9 Complex Numbers

In mathematics, the **complex numbers** are an extension of the **real numbers** obtained by adjoining an **imaginary unit**, denoted i, which satisfies

$$\mathrm{i}^2 = -1.$$

Complex numbers were first conceived and defined by the Italian mathematician *Gerolamo Cardano*, who called them *"fictitious"*, during his attempts to find solutions to cubic equations. The solution of a general cubic equation may require intermediate calculations containing the square roots of negative numbers, even when the final solutions are real numbers, a situation known as *casus irreducibilis*. This ultimately led to the **fundamental theorem of algebra**, which shows that with complex numbers, a solution exists to every polynomial equation of degree one or higher.

9.1 Definitions and operations

Let us first review our history of learning mathematics. The first set of numbers that we learned was the set of **positive integers** $\mathbb{Z}_+$, associated with which **addition** "+" and **subtraction** "−" were the earliest operations. At that time, we could not carry out $a - b$ if $a < b$. In other words, the set of positive integers is not **closed** under subtraction. This was a serious drawback for mathematics, so a larger set of numbers, namely negative numbers, was introduced. After supplementing positive integers with negative ones, the whole set of **integers** $\mathbb{Z}$ is then closed under both addition and subtraction. This supplementation was the first important extension of the number system.

Following addition and subtraction, we learned **multiplication** "×" and **division** "÷", still with the set of integers. In countless situations, when we divided an integer a by another integer b, the quotient $\dfrac{a}{b}$ was no longer an integer. This new drawback resulted in the introduction of an even larger set of numbers — **rational numbers** $\mathbb{Q}$. This was the second important extension of the number system.

The next extension occurred with the introduction of the n-th power "$\square^n$" and the n-th root "$\sqrt[n]{\square}$" of a rational number. An example of this could be that a number x was needed to satisfy $x^2 = 2$, thus $\sqrt{2}$ was introduced. These numbers are not rational, thus are called **irrational numbers**. Up till now, all numbers that we have been dealing with are called **real numbers** $\mathbb{R}$. However, the set of real numbers is still not closed under root extraction, for example, there exist no square roots of -1 in $\mathbb{R}$. For this purpose, we once again extend the number system by introducing the notion of **complex numbers** $\mathbb{C}$. A number x which satisfies the equation $x^2 + 1 = 0$ is then denoted i.

Complex numbers are written in the form

$$z = a + b\mathrm{i},$$

where a and b are real numbers, called the **real part** and the **imaginary part** of z, respectively, written as $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$. The number i is the **imaginary unit** satisfying $\mathrm{i}^2 = -1$.

When $b = 0$, $z = a$ is a **real number**; when $a = 0$ and $b \neq 0$, $z = b\mathrm{i}$ is called a **(pure) imaginary number**.

The set $\mathbb{R}$ is an ordered set: for any arbitrary real numbers x and y, exactly one of the following three relations holds: $x < y$; $x = y$; or $x > y$. However the set $\mathbb{C}$ is not ordered; for example, one cannot compare $2 - \mathrm{i}$ with $1 + 3\mathrm{i}$.

Two complex numbers are said to be **equal** if and only if both their real parts and imaginary parts are equal:

$$a_1 + b_1\mathrm{i} = a_2 + b_2\mathrm{i} \qquad \Longleftrightarrow \qquad \begin{cases} a_1 = a_2 \\ b_1 = b_2 \end{cases}$$

The set of complex numbers is closed under addition, subtraction, multiplication, division, power and root extraction. Associative, commutative and distributive laws of algebra for real numbers also work for complex numbers. With these laws, and the equation $\mathrm{i}^2 = -1$, we may carry out algebraic operations of complex numbers as follows:

- **Addition:** $(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$.

- **Subtraction:** $(a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i}$.

- **Multiplication:** $(a + b\mathrm{i}) \times (c + d\mathrm{i}) = ac + ad\mathrm{i} + bc\mathrm{i} + bd\mathrm{i}^2 = (ac - bd) + (ad + bc)\mathrm{i}$.

- **Division:** $\dfrac{a + b\mathrm{i}}{c + d\mathrm{i}} = \dfrac{(a + b\mathrm{i})(c - d\mathrm{i})}{(c + d\mathrm{i})(c - d\mathrm{i})} = \left(\dfrac{ac + bd}{c^2 + d^2}\right) + \left(\dfrac{bc - ad}{c^2 + d^2}\right)\mathrm{i}$, where $c + d\mathrm{i} \neq 0$.

Exercise 60.

1. Given $z = 3 + \mathrm{i}$ and $w = 1 - 2\mathrm{i}$, express the following in the form of $a + b\mathrm{i}$, where a and b are real numbers.
$$z + w\mathrm{i}, \qquad \frac{z}{w}, \qquad \frac{1 - \mathrm{i}}{w}, \qquad \frac{zw}{z + w}$$

2. Evaluate the following expressions.
$$\mathrm{Re}\,(2 + \mathrm{i})^2, \qquad \mathrm{Im}\,\frac{1}{2\mathrm{i}}, \qquad (1 + \sqrt{2}\mathrm{i})^3, \qquad \left(\frac{1 + \mathrm{i}}{\sqrt{2}}\right)^4, \qquad \left(\frac{1 - \sqrt{3}\mathrm{i}}{2}\right)^6$$

3. Given two complex numbers z and w, which of the followings are always true?
   (a) $\mathrm{Re}\,(z\mathrm{i}) = \mathrm{Im}\,z$
   (b) $\mathrm{Im}\,(w\mathrm{i}) = \mathrm{Re}\,w$
   (c) $\mathrm{Re}\,(z - 3w) = \mathrm{Re}\,z - 3\mathrm{Re}\,w$
   (d) $\mathrm{Im}\,(2z + w) = 2\mathrm{Im}\,z + \mathrm{Im}\,w$
   (e) $(\mathrm{Re}\,z) \times (\mathrm{Re}\,w) = \mathrm{Re}\,(zw)$
   (f) $\dfrac{\mathrm{Im}\,z}{\mathrm{Im}\,w} = \mathrm{Im}\left(\dfrac{z}{w}\right)$
   (g) $[(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2] \times [(\mathrm{Re}\,w)^2 + (\mathrm{Im}\,w)^2] = [\mathrm{Re}\,(zw)]^2 + [\mathrm{Im}\,(zw)]^2$

4. Given $(2 + \mathrm{i})(x + y\mathrm{i}) = 1 + 3\mathrm{i}$, where x and y are real, expand the multiplication then solve for x and y; then compare your answer with $\dfrac{1 + 3\mathrm{i}}{2 + \mathrm{i}}$ using the above division formula.

5. Let a, b and c be real numbers. Find a condition of the form $Aa + Bb + Cc = 0$, where A, B and C are integers, which ensures that $\dfrac{a}{1 + \mathrm{i}} + \dfrac{b}{1 + 2\mathrm{i}} + \dfrac{c}{1 + 3\mathrm{i}}$ is real.

6. Find two complex values of z such that $z^2 = \mathrm{i}$.

7. Find the square roots of the following numbers:
   (a) $-3 + 4\mathrm{i}$
   (b) $5 + 12\mathrm{i}$

8. Find four complex values of z such that $z^4 = -7 + 24\mathrm{i}$.

9.2 Modulus and conjugation

Since a complex number $z = x + y\mathrm{i}$ is uniquely specified by the ordered pair (x, y), the set of complex numbers is in one-to-one correspondence with points on a two-dimensional Cartesian plane, called the **complex plane** or **Argand diagram**, named after *Jean-Robert Argand*. In an Argand diagram, the axes are often referred to as the **real axis** and the **imaginary axis**. The representation of $z = x + y\mathrm{i}$ is called **Cartesian form** (or **rectangular form**, or **algebraic form**) of the complex number.

The **modulus** or **absolute value** of a complex number $z = x + y\mathrm{i}$ is defined as

$$|z| = \sqrt{x^2 + y^2}.$$

The modulus of complex numbers has the following properties (exactly the same as the absolute value of real numbers):

- **Non-negativity:** $\forall\ z \in \mathbb{C},\ |z| \geq 0.$
- **Positive-definiteness:** $\forall\ z \in \mathbb{C},\ |z| = 0 \iff z = 0.$
- **Multiplicativeness:** $\forall\ z, w \in \mathbb{C},\ |zw| = |z| \cdot |w|.$
- **Subadditivity:** $\forall\ z, w \in \mathbb{C},\ |z + w| \leq |z| + |w|.$

However, unlike real numbers, $|z|$ does *not* necessarily equal $\sqrt{z^2}$ if z is complex; the latter is even not well defined. Rather we have, if $z = a + b\mathrm{i} \in \mathbb{C}$,

$$|z|^2 = a^2 + b^2 = (a + b\mathrm{i})(a - b\mathrm{i}).$$

The above equation suggests the importance of the number $a - b\mathrm{i}$. Indeed, it is called the **(complex) conjugate** of the complex number $z = a + b\mathrm{i}$, written as $\overline{z}$ or z^*. z^* is the "reflection" of z in the real axis. Some identities concerning complex numbers and their conjugates are: $\forall z, w \in \mathbb{C}$,

- $(z \pm w)^* = z^* \pm w^*$;
- $(z \cdot w)^* = z^* \cdot w^*$;
- $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$;
- $(z^*)^* = z$;
- $z^* = z$ if and only if z is real;
- $z^* = -z$ if and only if z is either purely imaginary or zero;
- $\operatorname{Re} z = \frac{1}{2}(z + z^*)$;
- $\operatorname{Im} z = \frac{1}{2\mathrm{i}}(z - z^*)$;
- $|z| = |z^*|$;
- $|z|^2 = z \cdot z^*$.

Exercise 61.

1. Prove the above properties of modulus and conjugation, and draw diagrams wherever appropriate.
2. Assume $z = 2 - \mathrm{i}$ and $w = 4 + 5\mathrm{i}$, label the following points in an Argand diagram.
$$4z - (2 - \mathrm{i})w, \qquad z^* + w^*, \qquad \frac{z^* + 2}{w^* + \mathrm{i}}, \qquad (z + w)^*, \qquad \left(\frac{z + 2}{w - \mathrm{i}}\right)^*$$
3. Assume $z = 2 + 3\mathrm{i}$ and $w = 4 - \mathrm{i}$, evaluate the following.
$$\frac{|z + w|}{|z| + |w|}, \qquad (z - w)^*, \qquad \left|z^6 w^{-4}\right|, \qquad zw^* + z^* w, \qquad \left(z^2 + \frac{17}{w^2}\right)^*$$

In the geometrical point of view, the modulus of a complex number, $|z|$, denotes the distance between the point representing the number z and the origin in the Argand diagram. In general, the **distance** between two points Z and W in the Argand diagram is given by $ZW = |z - w|$, where the complex numbers z and w correspond to the points respectively.

Worked examples:

1. In the complex plane, points A and B represent $2 + \mathrm{i}$ and $-4 - \mathrm{i}$ respectively, identify the set of points Z such that $AZ = BZ$.

   Solution. Let Z represent $z = x + y\mathrm{i}$. The distances AZ and BZ are given by $AZ = z - 2 - \mathrm{i}$ and $BZ = z + 4 + \mathrm{i}$. Thus the equation $AZ = BZ$ reads

$$\begin{aligned} |z - 2 - \mathrm{i}| &= |z + 4 + \mathrm{i}| \\ |z - 2 - \mathrm{i}|^2 &= |z + 4 + \mathrm{i}|^2 \\ (z - 2 - \mathrm{i})(z - 2 - \mathrm{i})^* &= (z + 4 + \mathrm{i})(z + 4 + \mathrm{i})^* \\ (z - 2 - \mathrm{i})(z^* - 2 + \mathrm{i}) &= (z + 4 + \mathrm{i})(z^* + 4 - \mathrm{i}) \\ zz^* + (-2 + \mathrm{i})z + (-2 - \mathrm{i})z^* + 5 &= zz^* + (4 - \mathrm{i})z + (4 + \mathrm{i})z^* + 17. \end{aligned}$$

   Hence

$$\begin{aligned} (6 - 2\mathrm{i})z + (6 + 2\mathrm{i})z^* + 12 &= 0 \\ 6(z + z^*) - 2\mathrm{i}(z - z^*) + 12 &= 0 \\ 6(2x) - 2\mathrm{i}(2y\mathrm{i}) + 12 &= 0 \\ 3x + y + 3 &= 0 \end{aligned}$$

   The line $3x + y + 3 = 0$ is the perpendicular bisector of AB. You can check that any point on this line is equidistant form the points A and B.

   □

2. In the complex plane, points A and B represent 3 and $4\mathrm{i}$ respectively, identify the set of points Z such that $3AZ = 2BZ$.

   Solution. Let Z represent $z = x + y\mathrm{i}$. The distances AZ and BZ are given by $AZ = |z - 3|$ and $BZ = |z - 4\mathrm{i}|$. Thus the equation $3AZ = 2BZ$ reads

$$\begin{aligned} 3|z - 3| &= 2|z - 4\mathrm{i}| \\ (3|z - 3|)^2 &= (2|z - 4\mathrm{i}|)^2 \\ 9(z - 3)(z - 3)^* &= 4(z - 4\mathrm{i})(z - 4\mathrm{i})^* \\ 9(z - 3)(z^* - 3) &= 4(z - 4\mathrm{i})(z^* + 4\mathrm{i}) \\ 9(zz^* - 3z - 3z^* + 9) &= 4(zz^* + 4\mathrm{i}z - 4\mathrm{i}z^* + 16). \end{aligned}$$

   Hence

$$\begin{aligned} 5zz^* - 27(z + z^*) - 16\mathrm{i}(z - z^*) + 17 &= 0 \\ 5(x^2 + y^2) - 54x + 32y + 17 &= 0 \\ \left(x - \frac{27}{5}\right)^2 + \left(y + \frac{16}{5}\right)^2 &= 36 \\ \left|z - \left(\frac{27}{5} - \frac{16\mathrm{i}}{5}\right)\right| &= 6 \end{aligned}$$

   This suggests that the locus of the point Z is a circle with center $\left(\frac{27}{5} - \frac{16}{5}\mathrm{i}\right)$ and radius 6.

   □

Exercise 62.

1. Represent each of the following equations or inequalities in an Argand diagram.

$$|z-3|=2, \qquad |z-3\mathrm{i}|=|z+2|, \qquad |z+2-\mathrm{i}|<2, \qquad |2z+\mathrm{i}|\geq|2z-3|$$

$$\operatorname{Re} z+\operatorname{Im} z=2, \qquad \operatorname{Re} z=\sqrt{3}\operatorname{Im} z, \qquad zz^*=5, \qquad z-z^*=4\mathrm{i}$$

$$(\dagger) \qquad |3z|<|z+4|, \qquad \operatorname{Im} z+2\leq|z|, \qquad |z-\mathrm{i}|+|z+\mathrm{i}|>4, \qquad |z-2|-|z+2|=3$$

2. Given that $|z-4\mathrm{i}|=3$, find the greatest value of $|z+5|$.

3. Given that $|z-4|=|z+2\mathrm{i}|$, find the least value of $|z-\mathrm{i}|$.

4. Given that $|z-5|=1$, and $|w+1-\mathrm{i}|=|w+3+\mathrm{i}|$, find the least value of $|z-w|$.

5. If z and w are complex numbers, prove that $|z+w|^2+|z-w|^2=2(|z|^2+|w|^2)$, and interpret this equation geometrically.

6. If z and w are complex numbers, prove that zw^*+z^*w is a real number.

7. Two complex numbers, z and w, satisfying the inequalities $|z-3-2\mathrm{i}|\leq 2$ and $|w-7-5\mathrm{i}|\leq 1$. By drawing an Argand diagram, find the least possible value and the greatest possible value of $|z-w|$.

8. In the complex plane, points A, B and C represent the complex numbers i, $2-\mathrm{i}$ and $3+5\mathrm{i}$ respectively. Point D is equidistant to all these three points. Find the complex number that D represents.

9. In the complex plane, point Z represents the complex number z, identify the set of points Z such that $|z-2+\mathrm{i}|=\operatorname{Re} z$, describing the locus of Z geometrically.

10. (†) P is a point in an Argand diagram corresponding to a complex number z, and $|z+\mathrm{i}|+|z-\mathrm{i}|=4$. Without much calculation, prove $\sqrt{3}\leq|z|\leq 2$, and describe the locus of P geometrically.

9.3 The polar form

In addition to the Cartesian representation $z = x + y\mathrm{i}$, a complex number z can also be specified by the **polar form**. For a complex number z, its polar coordinates are the parameters r and φ, where $r = |z| \geq 0$ is its modulus (or absolute value), and $\varphi = \arg z$ is called its **argument**. The argument is defined as the angle subtended between the positive real axis and the line segment OZ in a counterclockwise direction in an Argand diagram, where Z is the point representing the complex number z.

It can be seen that

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \text{or} \qquad z = r(\cos\varphi + \mathrm{i}\sin\varphi).$$

These equations hold for all kinds of angles of φ.

For $z = 0$, its argument *may* be any angle. One convention is to set $\arg 0 = 0$, while another convention is to claim that $\arg 0$ is indeterminate.

For $z \neq 0$, the argument $\varphi = \arg z$ is unique modulo 2π, due to the periodicity of the sine and cosine functions. To obtain a unique representation, a conventional choice is to limit φ within the interval $(-\pi, \pi]$. This value of φ is sometimes called the **principal value** of $\arg z$, written $\operatorname{Arg} z$. (Throughout this course, the argument of a complex number always refers to its principal value, unless otherwise stated.) Thus any non-zero complex number z can be *uniquely* written in the polar form as $z = r(\cos\varphi + \mathrm{i}\sin\varphi)$, where $r = |z| > 0$ is its modulus and φ is the principal argument of its argument.

Given a non-zero complex number $z = x + y\mathrm{i}$ and its corresponding point Z on an Argand diagram, its argument φ can be determined as follows:

- If $x > 0$, Z lies in the right half plane, thus $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, which is exactly the range of inverse tangent function. So in this case, we have
$$\varphi = \tan^{-1}\tfrac{y}{x}.$$
- If $x = 0$, z is a pure imaginary, and Z is on the imaginary axis. Therefore we have
$$\varphi = \begin{cases} \frac{\pi}{2} & \text{if } y > 0; \\ -\frac{\pi}{2} & \text{if } y < 0. \end{cases}$$
- When $x < 0$, Z lies in the left half plane, thus φ is the range of $(\frac{\pi}{2}, \pi]$ or $(-\pi, -\frac{\pi}{2})$. Considering that $\tan^{-1}\frac{y}{x}$ would give an angle in the opposite direction of OZ, we have
$$\varphi = \begin{cases} \tan^{-1}\frac{y}{x} + \pi & \text{if } y \geq 0; \\ \tan^{-1}\frac{y}{x} - \pi & \text{if } y < 0. \end{cases}$$

In the polar form, a complex number z and its complex conjugate z^* have the same modulus, but opposite arguments:

$$[r(\cos\theta + \mathrm{i}\sin\theta)]^* = r(\cos\theta - \mathrm{i}\sin\theta) = r[\cos(-\theta) + \mathrm{i}\sin(-\theta)].$$

Exercise 63.

1. Plot the following complex numbers on an Argand diagram. For each of the numbers, find its modulus and argument.
$$2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right), \qquad \sqrt{2} - \sqrt{2}\mathrm{i}, \qquad -1 + \sqrt{3}\mathrm{i}, \qquad \sqrt{2} + 1 - \mathrm{i}, \qquad (\sqrt{2} - \sqrt{6} + 4\mathrm{i})^*$$
2. Find the argument of $z = 1 + \mathrm{i}$. Calculate $u = z^4$, $v = z^5$, and $w = z^6$, and find their arguments.
(†) Hence conjecture a relation between $\arg z$ and $\arg(z^k)$ for any $z \in \mathbb{C}$ and $k \in \mathbb{Z}^+$.
3. (†) Use your above conjecture to find the argument of $z = -(1 + \sqrt{3}\mathrm{i})^5$ and $w = \dfrac{4}{(\sqrt{3} - \mathrm{i})^7}$.

Geometrically, $\arg(z-w)=\varphi$ means that the point Z lies on the half-line starting from the point W and making an angle of φ with the positive real axis in the counterclockwise direction.

Worked examples:

1. If $\arg z = \frac{\pi}{4}$ and $\arg(z-3)=\frac{\pi}{2}$, find $\arg(z-6\mathrm{i})$.

   Solution. Let $z=x+y\mathrm{i}$. Since $\arg z=\frac{\pi}{4}$, the point Z lies on the half-line $y=x$ with $x>0$. Since $\arg(z-3)=\frac{\pi}{2}$, the translation from 3 to Z makes an angle of $\frac{\pi}{2}$ with the positive real axis, so the point Z lies on the half-line $x=3$ with $y>0$. These two half-lines meet at the point $(3,3)$, thus $z=3+3\mathrm{i}$. Hence $\arg(z-6\mathrm{i})=\arg(3-3\mathrm{i})=-\frac{\pi}{4}$. □

2. Given that $\arg(z-\mathrm{i})=\frac{3\pi}{4}$, and $|w+3|=1$, find least values of $|z-w|$.

   Solution. Let $z=x+y\mathrm{i}$ and $w=u+v\mathrm{i}$. Since $\arg(z-\mathrm{i})=\frac{3\pi}{4}$, we know that Z lies on the half-line $y=-x+1$, with $x<0$. $|w+3|=1$ means that W lies on the circle with radius 1 and center -3. The distance from the point -3 to this half-line is $2\sqrt{2}$, thus the least values of $|z-w|$ is $2\sqrt{2}-1$.

   □

Draw two diagrams to illustrate the above equations.

Exercise 64.

1. Use an Argand diagram to find, in the form $x+y\mathrm{i}$, the complex numbers which satisfy the following pairs of equations.
   (a) $\arg(z-\mathrm{i})=\frac{\pi}{2}$, and $|z+1|=5$
   (b) $\arg(2z+4\mathrm{i})=\pi$, and $\arg(z-2+3\mathrm{i})=\frac{5\pi}{6}$
   (c) $\arg(z+2+\mathrm{i})=\frac{\pi}{3}$, and $|z-1-2\mathrm{i}|=2$
   (d) $\arg(z-3-\mathrm{i})=-\frac{3\pi}{4}$, and $\arg(z+2)=-\frac{\pi}{4}$
2. Draw an Argand diagram, showing the set of points satisfying each of the following conditions. Use a solid line to show boundary points which are included, and a dotted line for boundary points which are not included.
$$\arg z=\frac{\pi}{5},\qquad \arg(z+1-\mathrm{i})=-\frac{2\pi}{5},\qquad 0<\arg z<\frac{\pi}{6},\qquad -\frac{\pi}{4}\le\arg(z+\mathrm{i})\le\frac{\pi}{4}$$
3. Draw an Argand diagram, and shade the region where both $|z-4\mathrm{i}|\le 3$ and $0<\arg(z+3)\le\frac{\pi}{4}$ are satisfied. Within this region, find
   (a) the least value of $\operatorname{Re} z$,
   (b) the least value of $\arg z$,
   (c) the value of arg where $\operatorname{Im} z$ is maximized.

By using trigonometric identities, it follows that the product of $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$ and $w = s(\cos\beta + \mathrm{i}\sin\beta)$ and be written as

$$\begin{aligned} zw &= r(\cos\alpha + \mathrm{i}\sin\alpha)\cdot s(\cos\beta + \mathrm{i}\sin\beta) \\ &= rs(\cos\alpha\cos\beta + \mathrm{i}\sin\alpha\cos\beta + \mathrm{i}\cos\alpha\sin\beta + \mathrm{i}^2\sin\alpha\sin\beta) \\ &= rs((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + \mathrm{i}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)) \\ &= rs(\cos(\alpha+\beta) + \mathrm{i}\sin(\alpha+\beta)). \end{aligned}$$

Therefore, it can be seen that the modulus of the product equals the product of the moduli, and that the argument of the product equals the sum of the arguments modulo 2π.

The rule of division can be obtained in a similar fashion for $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$ and $w = s(\cos\beta + \mathrm{i}\sin\beta)$:

$$\frac{z}{w} = \frac{r}{s}(\cos(\alpha-\beta) + \mathrm{i}\sin(\alpha-\beta)).$$

This is to say, the modulus of the quotient equals the quotient of the moduli, and the argument of the quotient equals the difference of the arguments modulo 2π.

Exercise 65.

1. If $z = -2-2\mathrm{i}$ and $w = \sqrt{3}-\mathrm{i}$, write zw and $\dfrac{z^*}{w}$ in the polar form.
2. Use division of complex numbers to find the angle AOB, where A and B represent the complex numbers $a = \sqrt{3}+\mathrm{i}$ and $b = 1-\mathrm{i}$ on an Argand plane, respectively.
3. Given $z = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$, $w = \cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}$ and $q = 4\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$, write the following in the polar form.
   (a) zw
   (b) $\dfrac{q}{w}$
   (c) zw^2q
   (d) wz^*
   (e) $\dfrac{4\mathrm{i}}{q^*}$
   (f) $\dfrac{z^*q^2}{w^3}$
4. (†) Identify the set of points on an Argand diagram for which $\arg\left(\dfrac{z-3}{z-4\mathrm{i}}\right) = \dfrac{\pi}{2}$.
   Also find the locus of Z representing the complex number z which satisfies: $\arg\left(\dfrac{z-3}{z-4\mathrm{i}}\right) = -\dfrac{\pi}{2}$.
5. Write $1+\sqrt{3}\mathrm{i}$ and $1-\mathrm{i}$ in the polar form, and express $\dfrac{(1+\sqrt{3}\mathrm{i})^4}{(1-\mathrm{i})^6}$ in the Cartesian form.
6. Given two complex numbers $p = 1+3\mathrm{i}$, $q = 5-\mathrm{i}$, and their corresponding points P and Q on an Argand plane, find two possible points R such that $\triangle PQR$ is an equilateral triangle.
7. Given two complex numbers $u = 2+\mathrm{i}$, $v = 4-\mathrm{i}$, and their corresponding points U and V on an Argand plane, find all possible points W such that $\triangle UVW$ is an isosceles right triangle.
8. In the Argand diagram, the points Q and A represent the complex numbers $4+6\mathrm{i}$ and $10+2\mathrm{i}$. If A, B, C, D, E, F are the vertices, taken in clockwise order, or a regular hexagon with center Q, find the complex numbers which represents B, D and F.

9.4 Solving quadratic equations

Solving quadratic equations of complex coefficients follows the same methods as in the real case.

Exercise 66.

Solve the following equations:

1. $z^2 + 2z + 3 = 0$
2. $z^2 + z + (1 - \mathrm{i}) = 0$
3. $(2 - \mathrm{i})z^2 + (3 + \mathrm{i})z - 5 = 0$
4. $z^2 - 2\mathrm{i}z - 1 = 5 - 12\mathrm{i}$
5. $(1 - \mathrm{i})z^2 + 2z + 4 = 0$
6. $(1 + 2\mathrm{i})z^2 + (-1 + 3\mathrm{i})z - 5\mathrm{i} = 0$
7. $(2 - \mathrm{i})z^2 + (4 + 3\mathrm{i})z + (-1 + 3\mathrm{i}) = 0$
8. $z^3 = 1$

If $p(z)$ is a polynomial with real coefficients and w is a non-real root of p, then w^* is also a root.

Proof. Let $p(z)$ be a polynomial of degree n:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

whose coefficients $a_n, a_{n-1}, \cdots, a_1$ and a_0 are all real numbers and $a_n \neq 0$. Then,

$$\begin{aligned}
(p(z))^* &= (a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)^* \\
&= (a_n z^n)^* + (a_{n-1} z^{n-1})^* + \cdots + (a_1 z)^* + a_0^* \\
&= a_n^* (z^*)^n + a_{n-1}^* (z^*)^{n-1} + \cdots + a_1^* z^* + a_0^* \\
&= a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + \cdots + a_1 z^* + a_0 && \text{(since the coefficients } a_k \text{ are all real)} \\
&= p(z^*).
\end{aligned}$$

Therefore, if $p(w) = 0$, $p(w^*) = (p(w))^* = 0^* = 0$. Thus non-real roots of the equation $p(z) = 0$ occur as conjugate pairs.

□

Exercise 67.

(†) Show that $z - 1 + \mathrm{i}$ is a factor of the polynomial $p(z) = z^4 - 3z^3 - 2z^2 + 10z - 12$.
Hence find all four roots of the equation $p(z) = 0$.

9.5 (‡) Fundamental theorem of algebra [EXTRA]

The **fundamental theorem of algebra** states that every non-constant polynomial with complex coefficients has at least one complex root. Therefore, it can be implied that every non-zero polynomial, with complex coefficients, has exactly as many complex roots as its degree, if each root is counted up to its multiplicity.

In spite of its name, there is no known purely algebraic proof of the theorem, and many mathematicians believe that such a proof does not exist. Besides, it is not fundamental for modern algebra; its name was given at a time in which algebra was basically about solving polynomial equations with real or complex coefficients.

Considering the readership of this course, almost all proofs of the theorem involve some analysis that is way too advanced. Therefore no proof is provided here.

9.6 (†) Exponential form [EXTRA]

A complex number in the polar form, $z = r(\cos\varphi + \mathrm{i}\sin\varphi)$, can also be written in the exponential form $z = r\mathrm{e}^{\mathrm{i}\varphi}$.

Proof. Let $f(\varphi) = \cos\varphi + \mathrm{i}\sin\varphi$ and $g(\varphi) = \mathrm{e}^{\mathrm{i}\varphi}$ be two complex-valued functions of a real variable. Assuming that the rules of exponents with real numbers also apply in the complex case, we may take derivatives of both functions with respect to φ:

$$\begin{aligned} f'(\varphi) &= -\sin\varphi + \mathrm{i}\cos\varphi = \mathrm{i}(\cos\varphi + \mathrm{i}\sin\varphi) = \mathrm{i}f(\varphi); \\ g'(\varphi) &= \mathrm{e}^{\mathrm{i}\varphi}\cdot\mathrm{i} = \mathrm{i}g(\varphi). \end{aligned}$$

Since f and g both satisfy the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}\varphi} = \mathrm{i}z,$$

and $f(0) = g(0) = 1$, the two functions must be identical for all values of φ.

□

In the exponential form, multiplication and division of complex numbers can be neatly expressed as

$$\left(r\mathrm{e}^{\mathrm{i}\alpha}\right)\cdot\left(s\mathrm{e}^{\mathrm{i}\beta}\right) = (rs)\mathrm{e}^{\mathrm{i}(\alpha+\beta)} \qquad \text{and} \qquad \frac{r\mathrm{e}^{\mathrm{i}\alpha}}{s\mathrm{e}^{\mathrm{i}\beta}} = \left(\tfrac{r}{s}\right)\mathrm{e}^{\mathrm{i}(\alpha-\beta)}.$$

It can be observed that these are consistent with the case of real numbers.

An extremely special case is, in the exponential form, when $\varphi = \pi$, we have $\mathrm{e}^{\pi\mathrm{i}} = -1$, or

$$\mathrm{e}^{\pi\mathrm{i}} + 1 = 0,$$

a quite neat equation called *Euler's identity.* This identity is considered by many to be remarkable for its mathematical beauty.

We conclude this section, hence the whole notes, by an application of the exponential form of complex numbers. We have seen, in an earlier topic, applying integration by parts twice to evaluate the following integrals:

$$\int \mathrm{e}^{ax}\sin(bx)\,\mathrm{d}x \qquad \text{and} \qquad \int \mathrm{e}^{ax}\cos(bx)\,\mathrm{d}x,$$

where a and b are real constants. Since $\mathrm{e}^{\mathrm{i}bx} = \cos(bx) + \mathrm{i}\sin(bx)$, we may substitute

$$\cos(bx) = \operatorname{Re}\left(\mathrm{e}^{\mathrm{i}bx}\right) \qquad \text{and} \qquad \sin(bx) = \operatorname{Im}\left(\mathrm{e}^{\mathrm{i}bx}\right).$$

Hence

$$\begin{aligned} \int \mathrm{e}^{ax}\sin(bx)\,\mathrm{d}x &= \int \mathrm{e}^{ax}\cdot\operatorname{Im}\left(\mathrm{e}^{\mathrm{i}bx}\right)\mathrm{d}x \\ &= \int \operatorname{Im}\left(\mathrm{e}^{ax+\mathrm{i}bx}\right)\,\mathrm{d}x \\ &= \operatorname{Im}\left(\int \mathrm{e}^{(a+b\mathrm{i})x}\,\mathrm{d}x\right) \\ &= \operatorname{Im}\left(\frac{\mathrm{e}^{(a+b\mathrm{i})x}}{a+b\mathrm{i}}\right) + C \\ &= \operatorname{Im}\left(\mathrm{e}^{ax}\cdot\frac{\cos(bx)+\mathrm{i}\sin(bx)}{a+b\mathrm{i}}\right) + C \\ &= \operatorname{Im}\left(\mathrm{e}^{ax}\cdot\frac{[a\cos(bx)-b\sin(bx)]+\mathrm{i}[a\sin(bx)+b\cos(bx)]}{a^2+b^2}\right) + C \\ &= \frac{\mathrm{e}^{ax}}{a^2+b^2}[a\sin(bx)+b\cos(bx)] + C. \end{aligned}$$

Exercise 68.

Work out a similar expression for the integral $\int \mathrm{e}^{ax}\cos(bx)\,\mathrm{d}x$ by using the exponential form of complex number.

[End of the notes. Last updated on October 19, 2021.]